Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Computational Physics 204 (2005) 399–403
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Short Note

Artificial boundary conditions for axisymmetric slow viscous flow

Joseph B. Keller *

Departments of Mathematics and Mechanical Engineering, Stanford University, Building 380, Room 382G, Stanford, CA 94305-2125, USA

Received 14 June 2004; received in revised form 15 October 2004; accepted 19 October 2004
Available online 19 November 2004

Abstract

Exact and approximate artificial boundary conditions are derived for computing axially symmetric Stokes flow around an axisymmetric body.

1. Introduction

To calculate numerically the flow of a fluid of infinite extent around a finite body, it is usual to surround the body by an artificial boundary B. The domain between the body surface S and the boundary B is the computational domain Ω. To complete the formulation of the problem in Ω, boundary conditions must be imposed at B. They are called artificial boundary conditions, because B is an artificial boundary. Such conditions for various partial differential equations are discussed by Givoli [3].

We shall derive exact artificial boundary conditions for Stokes flow around an axisymmetric body. This is a slow steady axisymmetric flow of a viscous incompressible fluid. In spherical polar coordinates, the stream function $\Psi(r,\theta)$ of such a flow satisfies the equation [6, p. 132, Eq. (1.9) and p. 133, Eq. (1.11)]

$$\left[\partial_r^2 + r^{-2}\sin\theta\partial_\theta\frac{1}{\sin\theta}\partial_\theta\right]^2\Psi(r,\theta) = 0. \tag{1.1}$$

Because (1.1) is linear, the solution can be separated into a known part representing a given incident flow and an unknown scattered part. The scattered part can be represented outside a sphere of radius R by the series

* Tel.: +1 650 723 0851; fax: +1 650 725 4066.
E-mail address: keller@math.stanford.edu.

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2004.10.027

$$\Psi(r,\theta)=\sum_{n=1}^{\infty}\left[A_n r^{2-n}+B_n r^{-n}\right]Q_n(\theta),\quad r\geqslant R. \tag{1.2}$$

Here, $Q_n(\theta)=N_nC_{-n}^{-1/2}(\cos\theta)$ is the Gegenbauer polynomial $C_{-n}^{-1/2}$ multiplied by the normalization constant N_n, while A_n and B_n are constants. We shall make use of (1.2) in deriving artificial boundary conditions. Since (1.1) is of fourth order, two boundary conditions must be imposed at B.

In Section 2, we derive exact boundary conditions. The derivation is similar to that of Keller and Givoli [5] for the Helmholtz equation, and it requires that B be a sphere. The resulting conditions are nonlocal since they involve integration over B. In Section 3, we present a sequence of local boundary conditions. They are analogous to those given by Bayliss and Turkel [1] for the Helmholtz equation. Engquist and Majda [2] derived similar local conditions for general linear differential equations.

2. Exact boundary conditions

To derive exact boundary conditions, we choose the artificial boundary B to be a sphere of radius R. Then at $r = R$ we take the inner product of both sides of (1.2) with $Q_j(\theta)$ and use the orthonormality of the Q_j to obtain

$$\int_0^{\pi}Q_j(\theta)\Psi(R,\theta)\sin\theta\,\mathrm{d}\theta=A_jR^{2-j}+B_jR^{-j},\quad j=1,2,\ldots \tag{2.1}$$

Next we differentiate (1.2) with respect to r and take the inner product of both sides of the resulting equation with Q_j to get

$$\int_0^{\pi}Q_j(\theta)\Psi_r(R,\theta)\sin\theta\,\mathrm{d}\theta=(2-j)A_jR^{2-j-1}-jB_jR^{-j-1},\quad j=1,2,\ldots \tag{2.2}$$

We solve (2.1) and (2.2) for A_jR^{2-j} and B_jR^{-j} to obtain

$$A_jR^{2-j}=\frac{j}{2}\int_0^{\pi}Q_j(\theta)\Psi(R,\theta)\sin\theta\,\mathrm{d}\theta+\frac{R}{2}\int_0^{\pi}Q_j(\theta)\Psi_r(R,\theta)\sin\theta\,\mathrm{d}\theta, \tag{2.3}$$

$$B_jR^{-j}=\frac{j-2}{2}\int_0^{\pi}Q_j(\theta)\Psi(R,\theta)\sin\theta\,\mathrm{d}\theta+\frac{R}{2}\int_0^{\pi}Q_j(\theta)\Psi_r(R,\theta)\sin\theta\,\mathrm{d}\theta. \tag{2.4}$$

Now we compute $\Psi_{rr}(R,\theta)$ and $\Psi_{rrr}(R,\theta)$ by differentiating (1.2) twice and thrice respectively and setting $r = R$. Then we use (2.3) and (2.4) for A_j and B_j in the resulting equations. In this way we get

$$\begin{aligned}\Psi_{rr}(R,\theta)=R^{-2}\sum_{n=1}^{\infty}Q_n(\theta)\bigg\{&\left[(2-n)(2-n-1)\frac{n}{2}+n(n+1)\frac{(n-2)}{2}\right]\int_0^{\pi}Q_n(\theta')\Psi(R,\theta')\sin\theta'\,\mathrm{d}\theta'\\&+[(2-n)(2-n-1)+n(n+1)]\frac{R}{2}\int_0^{\pi}Q_n(\theta')\Psi_r(R,\theta')\sin\theta'\,\mathrm{d}\theta'\bigg\}.\end{aligned} \tag{2.5}$$

$$\begin{aligned}\Psi_{rrr}(R,\theta)=R^{-3}\sum_{n=1}^{\infty}Q_n(\theta)\bigg\{&\left[(2-n)(2-n-1)(2-n-2)\frac{n}{2}-n(n+1)(n+2)\frac{(n-2)}{2}\right]\\&\cdot\int_0^{\pi}Q_n(\theta')\Psi(R,\theta')\sin\theta'\,\mathrm{d}\theta'+[(2-n)(2-n-1)(2-n-2)-n(n+1)(n+2)]\\&\times\frac{R}{2}\int_0^{\infty}Q_n(\theta')\Psi_r(R,\theta')\sin\theta'\,\mathrm{d}\theta'\bigg\}.\end{aligned} \tag{2.6}$$

Eqs. (2.5) and (2.6) are two exact nonlocal boundary conditions on $\Psi(r,\theta)$ at the artificial boundary $r = R$. They express the second and third r derivatives of Ψ in terms of Ψ and Ψ_r on $r = R$. They are exact because they are derived from the exact representation (1.2) in $r \geqslant R$.

We will now show that the problem in Ω with these conditions at B is well posed, and that its solution is exactly the restriction to Ω of the solution of the original problem in the infinite domain occupied by the fluid. To do so, we assume that the original problem is well posed, i.e., that it has a unique solution which depends continuously on the data given on the body surface S.

To prove that the problem in Ω does have a solution, we note that the restriction to Ω of the solution of the original problem is such a solution. To prove that it is unique, we suppose that there are two solutions in Ω. Each of them can be continued into the infinite domain of the fluid by (1.2) with A_j and B_j given by (2.3) and (2.4). The resulting solutions are continuous with three continuous derivatives at $r = R$, as a consequence of (1.2), (2.1), (2.2), (2.5) and (2.6). The fourth and higher radial derivatives can then be shown to be continuous by using (1.1). Thus these two solutions in the infinite domain are solutions of the original problem, which has a unique solution. Therefore the two solutions in Ω have to be identical.

Since (2.5) and (2.6) are exact, their accuracy does not depend upon R. Therefore they can be used with R small, which makes the computational domain Ω small. Furthermore they can be used with any numerical method in the computational domain. In particular, they can preserve the symmetry of the stiffness matrix in the finite element method.

Their disadvantages are:

(a) The artificial boundary B must be a sphere.
(b) They are nonlocal, so they involve integration over B, and they require the calculation of the $Q_j(\theta)$, but these are not significant difficulties.
(c) The sums in (2.5) and (2.6) must be truncated at a finite value N. This introduces errors $\delta\Psi_{rr}(R,\theta)$ in (2.5) and $\delta\Psi_{rrr}(R,\theta)$ in (2.6) associated with the higher modes $n > N$. They are given by

$$\delta\Psi_{rr}(R,\theta) = \mathrm{O}\big(R^{-(N+3)}\big), \quad \delta\Psi_{rrr}(R,\theta) = \mathrm{O}\big(R^{-(N+4)}\big) \quad \text{as } R \to \infty. \tag{2.7}$$

Thus the truncation error decreases as N increases and as R increases.

The truncated boundary conditions can be improved by modifying them as in Grote and Keller [4, Sections 3 and 4].

In Appendix A, uniqueness is proved for the problem in the computational domain Ω, with the truncated boundary conditions on B when the body is a sphere.

3. Local artificial boundary conditions

When the sums on the right-hand sides of (2.5) and (2.6) are omitted, which corresponds to truncation at $N = 0$, those equations become

$$\Psi_{rr}(r,\theta) = 0, \quad \Psi_{rrr}(r,\theta) = 0, \quad (r,\theta) \in B. \tag{3.1}$$

The conditions (3.1) are local artificial boundary conditions, which are approximations to the exact nonlocal conditions (2.5) and (2.6). As (2.7) shows, the errors in these two approximations are $\mathrm{O}(R^{-3})$ and $\mathrm{O}(R^{-4})$ respectively, which are small only if R is large.

A sequence of successively more accurate local boundary conditions can be formulated by introducing the operators L_N defined by

$$L_0 = \partial_r^2, \qquad L_N = \partial_r^2 \prod_{j=1}^{N} (j + r\partial_r), \quad N = 1, 2, \ldots. \tag{3.2}$$

Upon applying L_N to Ψ given by (1.2) we get

$$L_N\Psi(r,\theta) = \mathrm{O}[r^{-(N+3)}], \quad N = 0, 1, \ldots. \tag{3.3}$$

This suggests that we choose a value of $N \geqslant 0$ and any artifical boundary B, not necessarily a sphere, and impose the two conditions

$$L_N\Psi(r,\theta) = 0, \quad L_{N+1}\Psi(r,\theta) = 0 \qquad (r,\theta) \in B. \tag{3.4}$$

When $N = 0$ these conditions reduce to (3.1), but applied on any boundary B, not just on a sphere. They are analogous to the conditions of Bayliss and Turkel [1] for the Helmholtz equation.

The conditions (3.3) are local, they can be used on an artificial boundary B of any shape, and they become more accurate as r increases. However, for $N > 0$ they involve high order r derivatives. Their order can be reduced by using the differential equation (1.1) at the expense of introducing θ derivatives. The conditions with $N = 0$ involve only second and third derivatives, so they are the easiest to use.

Acknowledgement

I thank Professor Dmitri Papageorgiou for bringing this problem to my attention. This work was supported in part by AFOSR.

Appendix A. Uniqueness with the truncated boundary condition

We have already shown that the problem in the computational domain with the exact boundary conditions (2.5) and (2.6) is well posed. Now we consider the problem with these conditions truncated at N. We shall prove that the solution is unique for any $N \geqslant 0$ when the body is a sphere $r = a$ on which Ψ and Ψ_r are specified. Then the difference Ψ between two solutions satisfies (1.1) in $a \leqslant r \leqslant R$ with $\Psi = \Psi_r = 0$ at $r = a$ and the truncated forms of (2.5) and (2.6) at $r = R$.

Since the body is a sphere, the solution can be written in the form

$$\Psi(r,\theta) = \sum_{n=1}^{\infty} Q_n(\theta)u_n(r), \quad a \leqslant r \leqslant R. \tag{A.1}$$

The function $u_n(r)$ satisfies the fourth order ordinary differential equation

$$\left[\partial_r^2 - r^{-2}n(n+1)\right]^2 u_n(r) = 0, \quad a \leqslant r \leqslant R, \tag{A.2}$$

and at $r = a$ the boundary conditions

$$u_n(a) = \partial_r u_n(a) = 0. \tag{A.3}$$

At $r = R$ the boundary conditions for $n \geqslant N + 1$ are

$$\partial_r^2 u_n = 0, \quad \partial_r^3 u_n = 0 \qquad \text{at } r = R. \tag{A.4}$$

We multiply (A.2) by $u_n(r)$ and integrate the resulting equation from $r = a$ to $r = R$. After integrating by parts twice, we can write the result in the form

$$\int_a^R u_n\left[\partial_r^2 - r^{-2}n(n+1)\right]^2 u_n \,\mathrm{d}r = \int_a^R \left(\left[\partial_r^2 - r^{-2}(n+1)\right]u_n\right)^2 \mathrm{d}r + \left[u_n\left(\partial_r^3 u_n\right) - \left(\partial_r u_n\right)\left(\partial_r^2 u_n\right) + 2n(n+1)r^{-2}u_n^2\right]_a^R = 0. \tag{A.5}$$

In view of (A.3), the boundary term vanishes at $r = a$ for all values of n. Then by using (A.4) for $n \geqslant N + 1$ we get

$$\int_a^R \left(\left[\partial_r^2 - r^{-2}n(n+1)\right]u_n\right)^2 \mathrm{d}r + 2n(n+1)R^{-2}u_n^2(R) = 0. \tag{A.6}$$

From (A.6) we conclude that

$$\left[\partial_r^2 - r^{-2}n(n+1)\right]u_n(r) = 0, \quad a \leqslant r \leqslant R, \quad n \geqslant N+1. \tag{A.7}$$

The only solution of (A.7) satisfying (A.3) is $u_n(r) \equiv 0$.

By using the result that $u_n(r) \equiv 0$ for $n \geqslant N + 1$, we can write (A.1) as

$$\Psi(r,\theta) = \sum_{n=1}^{N} Q_n(\theta)u_n(r). \tag{A.8}$$

Now Ψ given by (A.8) satisfies the exact boundary conditions (2.5) and (2.6). Therefore the previous argument, in Section 2, shows that $\Psi \equiv 0$.

References

[1] A. Bayliss, E. Turkel, Radiation boundary conditions for wave-like equations, Commun. Pure Appl. Math. 33 (1980) 707–725.
[2] B. Engquist, A. Majda, Absorbing boundary conditions for the numerical simulations of waves, Math. Comput. 31 (1977) 629–652.
[3] D. Givoli, Numerical Methods for Problems in Infinite Domains, Elsevier, Amsterdam, 1992.
[4] M. Grote, J.B. Keller, On non-reflecting boundary conditions, J. Comput. Phys. 122 (1995) 231–243.
[5] J.B. Keller, D. Givoli, An exact non-reflecting boundary condition, J. Comput. Phys. 82 (1988) 172–192.
[6] W.E. Langlois, Slow Viscous Flow, Macmillan, New York, 1964.